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Functional Approach to Quantum Decoherence and the Classical Final Limit: The Mott and Cosmological Problems

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Decoherence and the approach to the classical final limit are studied in two similar cases: the Mott and the cosmological problems.

1. INTRODUCTION

One of the most important problems of theoretical physics in the recent years is the question: How and in what circumstances does a quantum system become classical [1]? In spite of great effort, the problem remains [2] and we are far from a complete understanding of many of its most fundamental features. In fact the most developed and sophisticated theory on the subject, decoherent histories, is not free of strong criticism [3].

For conceptual reason we will decompose the limit *quantum mechanics* → *classical mechanics* into two processes: *quantum mechanics* → *classical statistical mechanics* and *quantum statistical mechanics* → *classical mechanics*. There is almost unanimous opinion that the first process is produced by two phenomena:

(i) *Decoherence*, which in quantum systems restores the Boolean statistic typical of quantum mechanics.

(ii) *The limit* $\hbar \to 0$, which circumvents the uncertainty relation at the macroscopic level and allows one to find the classically behaved density functions via the Wigner integral.

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The second process is produced by these phenomena plus the localization (or production of correlation) phenomenon [4]. We will discuss the first process and briefly deal with the second at the end of each section.

The techniques to deal with the two phenomena related with the first process are not yet completely developed. One of the main problems is to find a proper and unambiguous definition of the so-called *pointer basis* where decoherence takes place.

Our contribution to solving this problem is based in the ideas of Segal [5, 6] and van Howe [7], reformulated by Antoniou *et al.* [8]. We have developed these ideas in refs. 9 and 10, where we have shown how the Riemann–Lebesgue theorem can be used to prove the destructive interference of the off-diagonal terms of the state density matrix yielding decoherence. Using this technique, we have found decoherence and the classical statistical equilibrium limit in simple quantum systems [4] where we have defined the final pointer basis in an unambiguous way.[3] In addition, the localization phenomenon appears in some cases.

This paper gives two examples of the method introduced in refs. 9 and 10, which we briefly review in Section 2, and used to find a general solution for the quantum-classical limit in ref. 4.

In Section 3, the method will be used to solve the problem known as "the Mott problem," after Sir Neville Mott, probably the first to study the subject [11]. Let us consider a radioactive nucleus, placed at the origin of coordinates O, inside a bubble chamber. We will see that classical radial trajectories appear in the chamber due to the emitted outgoing particles. We must explain this phenomenon. Theoretically we have a timeless structure, since the wave function $\psi(\mathbf{x})$ satisfies the eigenvalue equation

$$H\psi = \omega\psi \tag{1}$$

where

$$H = -\frac{1}{2M}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{L^2}{2Mr^2} + W(r) \tag{2}$$

with $W(r)$ the spherically symmetric potential barrier, which is the external wall of a potential well, and such that $\lim_{r\to\infty} W(r) = 0$. This will be our model for the nuclear forces [12]. Let us observe that there is no trace of the time in these equations.

The main problem is that, even if the symmetry is a spherical one, there is no *a priori* reason to explain why the classical trajectories are radial. We have the following list of facts to explain:

[3] The relation of our method to decoherent histories is studied in ref. 4. They turn out to be equivalent, but in our method the final pointer basis is more properly defined.

(i) Why does the nucleus quantum regime become the classical regime for the classical trajectories?

(ii) Why are these classical trajectories radial?

(iii) Why does the notion of time, necessary to explain the motion of the classical radial particles, appear in a timeless formalism, i.e., in that of Eq. (1).

(iv) Why are there only outgoing motions?[4]

At this stage we must observe that this set of problems is very similar to those of quantum cosmology, where, in fact, we must explain the outcome of the classical regime, the appearance of time, the nature of the classical trajectories in superspace, and the direction of the corresponding motion, i.e., the arrow of time. Then several of the most important quantum universe problems that we will discuss in the second example are already contained in our humble Mott model,

In Section 4 we consider our second example: the quantum cosmology problem, since the appearance of a classical universe in quantum gravity models is the cosmological version of the first problem. Then, decoherence must also appear in the universe [13].

In this paper, using our method, we will solve the two examples and we will find:

(i) Decoherence in all the dynamical variables and in a well-defined final pointer basis.

(ii) A final classical equilibrium limit, when $\hbar \to 0$, in such a way that the Wigner function $F_*^{\mathrm{W}} = \rho_*^{(\mathrm{cl})}$ of the asymptotic diagonal matrix ρ_* can be expanded as (e.g., in the cosmological problem)

$$\rho_*^{(\mathrm{cl})}([\mathbf{x}], [\mathbf{k}]) = \int p_{\{l\}[\mathbf{a}]}\rho_{\{l\}[\mathbf{a}_0]}^{(\mathrm{cl})}([\mathbf{x}], [\mathbf{k}])\, d\{l\}\, d[\mathbf{a}] \tag{3}$$

where $\rho_{\{l\}[\mathbf{a}_0]}^{(\mathrm{cl})}([\mathbf{x}], [\mathbf{k}])$ is a classical density strongly peaked[5] in a trajectory defined by the initial coordinates $\mathbf{a}$ and momenta l, and $p_{\{l\}[\mathbf{a}]}$ is the probability of each trajectory. As essentially the limit of quantum mechanics is not classical mechanics, but classical statistical mechanics, this is our final result: the density matrix is translated in a classical density, via a Wigner function,

[4] But let us complete the first example by discussing the role of the global atmosphere of the bubble chamber in our problem. This role is nonessential and really just incidental. In fact, if there were no bubble chamber, at least for $r \to \infty$ the classical trajectories would also be radial, since in this case, any small detector located far enough from the origin would find radial motions. Furthermore, it would be very difficult to believe that these radial motions are produced only by the small detector. Then the radial motion exists even if there is no bubble chamber and it is a mistake to consider that it is the bubble chamber that, acting as an environment, produces the radial structure. But we must remark that there is a measurement in both cases and therefore the measurement process is essential.

[5] Precisely: peaked as allowed by the uncertainty principle.

and it is decomposed as a sum of densities peaked around all possible classical trajectories, each one of these densities weighted by its own probability.[6]

Thus our quantum density matrix behaves in its classical limit as a statistical distribution among a set of classical trajectories. Similar results, for the cosmological case, are obtained in refs. 14 and 15.

We will return to these conclusions in Section 5.

2. REVIEW OF THE METHOD

In order to go from the quantum to the classical statistical regime two new properties must appear:

(i) *Decoherence*: The density matrices, which contain quantum interference terms, must become diagonal, in such a way that these interferences are suppressed. Then the quantum way to find probabilities of exhaustive events (i.e., adding the corresponding amplitude and computing the norm) becomes the classical way: just adding the probabilities.

(ii) *The limit* $\hbar \to 0$: The positions and the momenta (or more generally canonically conjugate dynamical variables) can be defined as allowed by the uncertainty principle, but large scales (i.e., when $\hbar \to 0$) allow us to consider both the position and the momentum as independent dynamical variables, as in classical mechanics. Of course this independence is the essential property for finding a classical behavior.

These two closely related properties introduce classical statistical behavior in the quantum formalism. Let us begin with decoherence.

We are only interested in scattering states with continuous spectrum, e.g., for the first example, the Mott problem, the radial outgoing particles are described by these states. To obtain the Hamiltonian eigenbasis of the Hilbert space $\mathcal{H}$ we can consider, e.g., the Hamiltonian (2) and construct the Lippmann–Schwinger basis $\{|\omega+\rangle\}$ [27] (really $|\omega, l, n+\rangle$), where ω is partially discrete and partially continuous, e.g., it will have some ω_0 for the ground state and a continuous ω for the scattering states, but for the moment we will consider only the continuous index ω (since we are only interested in these states[7]). Of course, we can as well use $\{|\omega-\rangle\}$. Then the Hamiltonian can be diagonalized as

$$H = \int_0^\infty \omega |\omega+\rangle\langle\omega+|\, d\omega \tag{4}$$

From this expression we can deduce that the most general observable that we can consider reads

[6] After the classical statistical regime is reached correlations will eventually produce a pure classical regime.

[7] Bound states are considered in refs. 9 and 10.

$$O = \int_0^\infty O_\omega |\omega+\rangle\langle\omega+| \, d\omega + \int_0^\infty \int_0^\infty O_{\omega\omega'} |\omega+\rangle\langle\omega'+| \, d\omega \, d\omega' \qquad (5)$$

where the functions O_ω, $O_{\omega\omega'}$ are regular, namely, the most general observable must have a singular component (the first term of the rhs of the last equation) and a regular part (the second term). If the singular term were to be missing, the Hamiltonian would not belong to the space of the chosen observables [9]. Then $O \in \mathbb{O} \subset \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$ since some extra conditions must be added [9]. This space has the basis $\{|\omega), |\omega, \omega')\}$:

$$|\omega) = |\omega+\rangle\langle\omega+|, \; |\omega, \omega') = |\omega+\rangle\langle\omega'+| \qquad (6)$$

The regular quantum state ρ is measured by the observables just defined, computing the mean values of these observable in the quantum states $\langle O\rangle_\rho = \mathrm{Tr}(\rho O)$ [16]. These mean values can be considered as linear functionals ρ on the vectors O. Then the notion of state can be generalized to any linear functional over $\mathbb{O}$, which we can call $(\rho|O)$ [6]. In this way not only regular states, but singular states can be defined. Moreover, as ρ must be normalized, self-adjoint, and positive definite, $\rho \in \mathcal{S} \subset \mathbb{O}'$, where $\mathcal{S}$ is a convex set contained in $\mathbb{O}'$ [9, 10]. The basis of $\mathbb{O}'$ is $\{(\omega|, (\omega, \omega'|\}$. These states are defined as functionals by the equations

$$(\omega|\omega') = \delta(\omega - \omega'), \qquad (\omega, \omega''|\omega', \omega''') = \delta(\omega - \omega')\delta(\omega'' - \omega''') \quad (7)$$

Therefore a generic quantum state reads

$$\rho = \int_0^\infty \rho_\omega (\omega| \, d\omega + \int_0^\infty \int_0^\infty \rho_{\omega\omega'} (\omega, \omega'| \, d\omega \, d\omega' \qquad (8)$$

where $\rho_\omega \geq 0$, $\rho_{\omega\omega'} = \rho^*_{\omega'\omega}$. The states such that $\rho_\omega \neq \rho_{\omega\omega}$ will be called *generalized states* [9] and those such that $\rho_\omega = \rho_{\omega\omega}$ are the usual regular mixed or eventually pure states. To continue, even if the time is not strictly defined [since we have only Eq. (1)], let us postulate that there is a symmetry in the system with a symmetry group e^{-iHt}. At this level of reasoning this is a global fact imposed by the structure of the universe where we suppose the model is immersed (we will come back to this problem in Section 3.3.1; of course this will also be the case in the cosmological example because the problem of time definition is the same). Then the time evolution of the quantum state ρ reads

$$\rho(t) = \int_0^\infty \rho_\omega(\omega| \, d\omega + \int_0^\infty \int_0^\infty \rho_{\omega\omega'} e^{i(\omega-\omega')t}(\omega, \omega'| \, d\omega \, d\omega' \tag{9}$$

As in the statistical level we are considering we only measure mean values of observables in quantum states, i.e.,

$$\begin{aligned} \langle O \rangle_{\rho(t)} &= (\rho(t)|O) \\ &= \int_0^\infty \rho_\omega Q_\omega \, d\omega + \int_0^\infty \int_0^\infty \rho_{\omega\omega'} O_{\omega\omega'} e^{-i(\omega-\omega')t} \, d\omega \, d\omega' \end{aligned} \tag{10}$$

using the Riemann–Lebesgue theorem, we obtain the weak limit, for all $O \in \mathbb{O}$, $\rho \in \mathcal{S}$:

$$\lim_{t\to\infty} \langle O \rangle_{\rho(t)} = \langle O \rangle_{\rho_*} \tag{11}$$

where we have introduced the diagonal equilibrium state

$$\rho_* = \int_0^\infty \rho_\omega(\omega| \, d\omega \tag{12}$$

Therefore, in a weak sense we have

$$W \lim_{t\to\infty} \rho(t) = \rho_* \tag{13}$$

Thus, any quantum state goes to an equilibrium diagonal state weakly, and that will be the result if we observe and measure the system evolution with *any possible observable of space* $\mathbb{O}$, e.g., for the first example, with a global bubble chamber (or with any of the local small detectors of footnote 4 in Section 1). Then, from the observational point of view we have decoherence of the energy levels, even if from the strong limit point of view the off-diagonal terms never vanish, they just oscillate [Eq. (9)]. So, from now on, as we will consider the matrix $\rho(t)$ for large t, e.g., for the first example, far away from the nucleus, the relevant state will be ρ_*, a diagonal state in the energy.

Some observations are in order:

(i) The real existence of the two singular parts introduced above is assured by the nature of the problem. The singular part of the observables is just a necessary generalization of the singular part of the Hamiltonian, which is completely singular [Eq. (4)]. The singular part of the states is the final state for decoherence [Eq. (12)]. $(\rho|O)$ is just the natural generalization to the continuous of the trace of the product of two finite-dimensional matrices [Eq. (10)].

(ii) To fully understand the phenomenon, it is necessary to use generalized states. Therefore, any explanation only based on pure wave function states or mixed states is incomplete.[8]

Having established the decoherence in the energy, we must consider the decoherence in the other dynamical variables. Before going to the model, let us study the general case (we will repeat this reasoning in Sections 2.2 and 4.2, with the notation corresponding to each case). The diagonal singular component of Eq. (9) (which is equal to ρ_*) is time independent, therefore it is impossible that a different decoherence process takes place in this component to eliminate the off-diagonal terms in the other dynamical variables. Therefore, the only thing to do is to try to find if there is a basis where these diagonal terms vanish at any time and therefore there is a perfect and complete decoherence. For $t \to \infty$ this basis in fact exists and it is known as the *final pointer basis*.

Let $\{H, O_1, \ldots, O_N\}$ be the usual complete set of commuting observables (CSCO) that we are using to make our calculations and $\{|\omega, m_1, \ldots, m_N+\rangle\}$ (which we will simply call $\{|\omega, m_1, \ldots, m_N\rangle\}$ from now on) the corresponding eigenbasis. Then introducing the new indices in Eq. (12), we obtain for the equilibrium diagonal state

$$\rho_* = \int \sum_{m_1,\ldots,m_N,m'_1,\ldots,m'_N} \rho^{(\omega)}_{m_1,\ldots,m_N,m'_1,\ldots,m'_N}(\omega, m_1, \ldots, m_N, m'_1, \ldots, m'_N|\, d\omega \tag{14}$$

From what we have said under Eq. (8) we have

$$(\rho^{(\omega)}_{m_1,\ldots,m_N,m'_1,\ldots,m'_N})^* = \rho^{(\omega)}_{m'_1,\ldots,m'_N,m_1,\ldots,m_N} \tag{15}$$

Therefore this matrix can be diagonalized and there is a basis $\{(\omega, l_1, \ldots, l_N|\}$ where the matrix ρ_* reads

[8] Now an incidental question for the first example would be: which energies? In the first example the potential well, surrounded by the barrier, $W(r)$ originates unstable levels with energy ω_n which decay with a decaying time γ_n^{-1}. Inside the well there are oscillating waves that can be used to fulfill the boundary conditions at $r = 0$. When these waves arrive at the barrier they are partially reflected and transmitted [12]. The states that tunnel the barrier appear with an energy that peaks strongly at ω_n [9, 17]. Therefore the energy of the outgoing particles is in the form of energy packets $\sim\delta(\omega - \omega_n)$, and therefore could be essentially labeled with n. But taking into account that the energy spectrum is really a continuous one, we will always refer to it as ω.

$$\rho_* = \int \sum_{l_1,\ldots,l_N} \rho^{(\omega)}_{l_1,\ldots,l_N}(\omega, l_1, \ldots, l_N|\, d\omega \tag{16}$$

Now we can define the observables

$$P_i = \int \sum_{l_1,\ldots,l_N} P^{(i,\omega)}_{l_1,\ldots,l_N}(\omega, l_1, \ldots, l_N|\, d\omega \tag{17}$$

and the CSCO $\{H, P_1, \ldots, P_N\}$, where the singular component ρ_* is diagonal in the dynamical variables corresponding to the observables $P_1, \ldots, P_N$ from the very beginning. This is the *final pointer basis* where there is perfect decoherence. The final pointer basis is therefore defined by the dynamics of the model and by the quantum state considered, in complete agreement with the literature on the subject. The classical statistical limit will be complete if we transform all these equations via a Wigner integral as we will in each example using the corresponding notation.

3. THE MOTT PROBLEM

3.1. Decoherence

After these general considerations let us now go to our first problem. The Hamiltonian of Eq. (2) can be decomposed as

$$H = H_0 + V \tag{18}$$

$$H_0 = -\frac{1}{2M}\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{L^2}{2Mr^2} \tag{19}$$

$$V = W(r) \tag{20}$$

allowing the definition of the Lippmann–Schwinger basis.

Its essential property is its spherical symmetry. We will see how this symmetry leads directly to the result above, avoiding the diagonalization procedure used in Section 2. In order to conserve this symmetry, it is necessary that the nucleus prepares only spherically symmetric states. Thus, we can foresee that the CSCO corresponding to the pointer basis must contain the generator of angular rotation, so it must be $\{H, L^2, L_z\}$. The pointer basis must then be the usual $\{|\omega, l, m\rangle\}$ basis.

In fact, let us consider an initial state functional ρ_0 with spherical symmetry. If $\overline{\mathbb{L}}$ is the generator of the density rotations, a rotation of the state by an angle $\overline{\varphi}$ gives $\rho_0 = \exp\{-i\overline{\mathbb{L}} \cdot \overline{\varphi}\}\rho_0$, or equivalently

$$(\exp\{-i\overline{\mathbb{L}} \cdot \overline{\varphi}\}\rho_0|O) = (\rho_0|\,\exp\{i\overline{L} \cdot \overline{\varphi}\}O\,\exp\{i\overline{L} \cdot \overline{\varphi}\})$$

for any observable $O \in \mathbb{O}$. In the last expression, $\overline{L}$ is the angular momentum operator. From the last equation we obtain

$$(\bar{\mathbb{L}}\rho_0|O) = (\rho_0|[\bar{L}, O]) = 0 \tag{21}$$

The observable O has the form

$$O = \sum_{lm}\sum_{l'm'}\int d\omega\, O_{lm,l'm'}(\omega)|\omega lm\rangle\langle\omega l'm'| + \sum_{lm}\sum_{l'm'}\iint d\omega\, d\omega'\, O_{lm,l'm'}(\omega,\omega')|\omega lm\rangle\langle\omega' l'm'| \tag{22}$$

and therefore Eq. (21) gives

$$(\rho_0|[\bar{L}, |\omega lm\rangle\langle\omega' l'm'|]) = 0 \tag{23}$$

These equations are equivalent to

$$(\rho_0|[L_z, |\omega lm\rangle\langle\omega' l'm'|]) = 0$$
$$(\rho_0|[L_\pm, |\omega lm\rangle\langle\omega' l'm'|]) = 0, \qquad L_\pm = L_x \pm L_y \tag{24}$$

Taking into account that

$$L_z|l, m\rangle = m|l, m\rangle, \qquad L_\pm|l, m\rangle = \sqrt{l(l+1) - m(m\pm 1)}|l, m\pm 1\rangle \tag{25}$$

we obtain

$$0 = (m - m')(\rho_0||\omega, l, m\rangle\langle\omega', l', m'|)$$
$$0 = \sqrt{l(l+1) - m(m\pm 1)}(\rho_0||\omega, l, m\pm 1\rangle\langle\omega', l', m'|) - \sqrt{l'(l'+1) - m'(m'\mp 1)}(\rho_0||\omega, l, m\rangle\langle\omega', l', m'\mp 1|) \tag{26}$$

both for $\omega = \omega'$ and $\omega \neq \omega'$. These equations give

$$(\rho_0||\omega, l, m\rangle\langle\omega', l', m'|) = \rho_l^0(\omega, \omega')\,\delta_{ll'}\,\delta_{mm'} \tag{27}$$
$$(\rho_0||\omega, l, m\rangle\langle\omega, l', m'|) = \rho_l^0(\omega)\,\delta_{ll'}\,\delta_{mm'}$$

Thus any symmetric ρ_0 is diagonal in l and m. We will repeat the conclusion of the previous section and see, in this particular case, how the time evolution produces the diagonalization: namely $\rho_l^0(\omega, \omega')$ will vanish when $t \to \infty$. As in Eq. (10), the time evolution reads

$$\langle O\rangle_{\rho_t} = (\rho_t|O) = (\rho_0|e^{iHt}Oe^{-iHt})$$
$$= \int d\omega \sum_{l=0}^{\infty}\rho_l^0(\omega)\sum_{m=-l}^{+l}O_{lm,lm}(\omega)$$
$$+ \int d\omega \int d\omega' \sum_{l=0}^{\infty}\rho_t^0(\omega,\omega')\exp\{i(\omega-\omega')t\}$$

$$\times \sum_{m=-l}^{+l} O_{lm,lm}(\omega, \omega') \tag{28}$$

The Riemann–Lebesgue theorem can be used in this expression to obtain the "final" state

$$\langle O \rangle_{\rho_*} = (\rho_* | O) = \lim_{t \to \infty} (\rho_t | O) = \int d\omega \sum_{l=0}^{\infty} \rho_l^0(\omega) \sum_{m=-l}^{+l} O_{lm,lm}(\omega) \tag{29}$$

where $\rho_l^0(\omega, \omega')$ has disappeared. If we define, in analogy to Eqs. (6) and (7), the functional $(\omega, l, m|$ acting on an observable O of the form given in Eq. (22) by $(\omega, l, m|O) = O_{lm,lm}(\omega)$, we can give the following expression for the asymptotic form of the state:

$$(\rho_*| = W \lim_{t \to \infty} (\rho_t | O) = \int d\omega \sum_{l=0}^{\infty} \rho_l^0(\omega) \sum_{m=-l}^{+l} (\omega, l, m| \tag{30}$$

Of course $(\rho_*|$ is also spherically symmetric since it is symmetric in l and m. We do not address in this paper the mechanism used by the nucleus to prepare the states in such a spherically symmetric way. But we are just studying the case where all the elements of the nucleus are spherically symmetric, and also the quantum states involved, because we are precisely trying to explain the breaking of this symmetry and the appearance of the radial structure. Then, it is clear that the only possibility is to begin with a spherically symmetric structure and with states that satisfy the above equations.

Therefore:

(i) The origin of the decoherence in the energy is the time evolution.

(ii) The origin of the decoherence in the angular variables is the preparation of the quantum state, which is spherically symmetric in our model. This symmetry is preserved by the time evolution.

Let us observe that, as our model is spherically symmetric, any CSCO $\{H, \mathbf{L}^2, L_z\}$, for any arbitrary z axis, will correspond to the final pointer basis. But if the symmetry were cylindrical along the axis z, as the center of an angular coordinate φ with generator L_z, the only CSCO related to a final pointer basis for the cylindrical quantum states would be $\{H, p_z, L_z\}$. So we see how the symmetry of the equation and the states defines the final pointer bases and their number.

3.2. The Limit $\hbar \to 0$ and the Classical $\rho_*^{(cl)}(q, p)$

Let us now compute the classical analogue of ρ_*, as promised at the end of Section 2. We will prove that the distribution function $\rho_*^{(W)}(q, p)$ that corresponds to the density matrix ρ_* via the Wigner integral [18] is simply a function of the classical constant of the motion, in our case $H(q, p)$, $\mathbf{L}^2(q, p)$, $L_z(q, p)$; precisely,

$$\rho_*^{(W)}(q, p) = \rho_*(H(q, p), \mathbf{L}^2(q, p), L_z(q, p)) \tag{31}$$

To simplify the demonstration, let us consider only the constant $H(q, p)$. From Eq. (12) we have

$$\rho_* = \int \rho_*(\omega)(\omega|\, d\omega \tag{32}$$

So we must compute

$$\rho_\omega^{(W)}(q, p) = \pi^{-1} \int (\omega \| q + \lambda\rangle\langle q - \lambda|) e^{2ip\lambda}\, d\lambda \tag{33}$$

We know, from ref. 9, Section IIC, that the characteristic property of $(\omega|$ is

$$(\omega|H^n) = \omega^n \tag{34}$$

Using the relation between quantum and classical inner products of operators [ref. 18, Eq. (2.13)], we deduce that the characteristic property of $\rho_\omega^{(W)}(q, p)$ is

$$\int \rho_\omega^{(W)}(q, p)[H(q, p)]^n\, dg\, dp = \omega^n + O(\hbar) \tag{35}$$

for any natural number n. From now on we will work in the limit $\hbar \to 0$ and therefore all the $O(\hbar)$ will disappear. Thus $\rho_\omega^{(\mathrm{cl})}(q, p)$ must be

$$\rho_\omega^{(W)}(q, p) = \delta(H(q, p) - \omega) > 0 \tag{36}$$

which turns out to be a distribution, namely a functional as ρ_*. Therefore, going back to Eq. (32) and since the Wigner relation is linear, we have

$$\begin{aligned} \rho_*^{(W)}(q, p) &= \int \rho_*(\omega)\rho_\omega^{(W)}(q, p)\, d\omega \\ &= \int \rho_*(\omega)\delta(H(q, p) - \omega)\, d\omega = \rho_*(H(q, p)) > 0 \end{aligned} \tag{37}$$

QED

Generalizing this reasoning, we can prove Eq. (31). Moreover the generalized equation (37) reads

$$\rho_*^{(W)}(q, p) = \int \sum_{l,m} \rho_*(\omega, l, m)\, \rho_{\omega,l,m}^{(W)}(q, p)\, d\omega \tag{38}$$

where $\rho_*(\omega, l, m) = \rho_l^0(\omega)$ of Eq. (27),[9] and $\rho_{\omega,l,m}^{(W)}(q, p)$ reads

$$\rho_{\omega,l,m}^{(W)}(q, p) = \delta(H(q, p) - \omega)\delta(\mathbf{L}^2(q, p) - l(l + 1))\delta(L_z(q, p) - m) \tag{39}$$

and can be interpreted as the state where ω, l, m are well defined and the corresponding classical canonically conjugated variables completely undefined since $\rho_{\omega,l,m}^{(W)}$ is not a function of these variables. Then there is a complete coincidence with the result of the previous section. From the last two equations we obtain (31) as promised. Now all the classical canonically conjugate variables of the momenta H, $\mathbf{L}^2$, L_z do exist since they can be found by solving the corresponding Poisson brackets differential equations.[10] But as the momenta H, $\mathbf{L}^2$, L_z, which we will call generically l, are also constants of the motion, we have $\dot{l} = -\partial H/\partial a = 0$, where a is the coordinate canonically conjugate to l, so H is just a function of l, and

$$\dot{a} = \frac{\partial H(l)}{\partial l} = \varpi(l) = \text{const} \tag{40}$$

so

$$a = \varpi(l)t + a_0 \tag{41}$$

These are the classical motions corresponding to the motion of the wave packet of the previous subsection. As in this section l is completely defined and a_0 completely undefined due to spherical symmetry, in such a way that the motions represented in the last equation homogeneously fill the surface $l = \text{const}$ (really $H = \text{const}$, $\mathbf{L}^2 = \text{const}$, $L_z = \text{const}$ for our case), namely the usual classical torus of phase space.

Then, Eq. (38) can be considered as the expansion of $\rho_*^{(\text{cl})}(q, p)$ in the classical motion just described, contained in $\rho_{n,l,m}^{(\text{cl})}(q, p)$, each one with a probability $\rho_*(n, l, m)$.

Summing up:

(i) We have shown that the density matrix $\rho(t)$ evolves to a diagonal density matrix ρ_*.

(ii) This density matrix has $\rho_*^{(\text{cl})}(q, p)$ as its corresponding classical density.

[9] Even if, for symmetry reasons, m is absent as an index of $\rho_l^0(\omega)$, we have introduced this index in $\rho_*(\omega, l, m)$ for two reasons: (i) It may be present in a more general case (such as the cosmological one), and (ii) it is present in $\rho_{\omega,l,m}^{(\text{cl})}$.

[10] On the contrary, H and L^2 do not have quantum canonically conjugated dynamical variables since their spectra are bounded from below.

(iii) This classical density can be decomposed into classical motions where H, $\mathbf{L}^2$, and L_z remain constant, answering question (i) of the introduction.

(iv) Finally, for $r \to \infty$ the classical analogue of Eq. (18),

$$H = \frac{1}{2M} p_r^2 + \frac{\mathbf{L}^2}{2Mr^2} + W(r) \tag{42}$$

shows that in the classical motions we can consider $\mathbf{L}^2 = 0$ when $r \to \infty$.[11] Thus the classical motions far from the nucleus can be considered as radial, answering question (ii) of the introduction.

Finally, as

$$\int \delta(\theta - \theta^{(0)})\delta(\varphi - \varphi^{(0)})\, d\theta^{(0)}\, d\varphi^{(0)} = 1 \tag{43}$$

where θ and φ are the usual polar coordinates. Since far from the nucleus $l = m = 0$, we can write Eq. (38) as

$$\rho_*^{(W)}(q, p) = \int \rho_*(\omega, 0, 0)\delta(H(q, p) - \omega)\delta(\theta - \theta^{(0)})\delta(\varphi - \varphi^{(0)})\, d\theta^{(0)}\, d\varphi^{(0)}\, d\omega \tag{44}$$

where $\delta(H(q, p) - \omega)\delta(\theta - \theta^{(0)}\delta(\varphi - \varphi^{(0)})$ can be considered as the classical density of the classical particles with radial motions defined by the energy ω and the angles $\theta^{(0)}$ and $\varphi^{(0)}$. The final classical equilibrium density is thus decomposed in the densities of radial classical trajectories, as in Eq. (3).

So finally we have an isotropic ensemble of particles in radial motion. This is the classical statistical limit of our wave function. We have gone from quantum mechanics to classical statistical mechanics [16]. To single out any one of the trajectories to obtain a single classical motion is clearly impossible at the statistical mechanics level since it would break the spherical symmetry of the initial wave function. But the statistical state is composed of single radial motions at the classical level, as any population is composed of individuals.

3.3. Localization and Correlations

These phenomena will appear for appropriate potential and initial conditions [4]. If $W(r) = 0$, we know that the wave packet will spread, so the real $W(r)$ must be such that the eventual spreading will be as slow as necessary in order to see the radial trajectories in the bubble chamber.

[11] A close study of the initial conditions, which necessarily are located near the center of the nucleus, will improve the demonstration of this result. For the sake of conciseness we do not include these considerations.

3.4. Discussion and Comments

3.4.1. The Time

We have postulated the global existence of time in Section 2.1. This postulate was motivated by the fact that this is a global feature of the universe. But, in order to mimic the cosmological case of the second example, let us imagine for a moment that our model can be considered as a model of the whole universe.[12] Then it would be impossible to postulate the existence of time based on an exterior global structure. A way to solve this problem would be to postulate decoherence through Eq. (12), namely the existence of some kind of evolution with a final diagonal state which corresponds to the fact that the universe really ends in a classical state. Then we can define the time over the classical trajectory labeled by ω and l via Eq. (42) as

$$M \frac{dr}{dt} = \sqrt{2M\left[\omega - \frac{l(l+1)}{2Mr^2} - W(r)\right]} \tag{45}$$

i.e.,

$$t = \frac{1}{\sqrt{2}} \int_0^r \frac{dr}{\sqrt{\omega - l(l+1)/(2Mr^2) - W(r)}} \tag{46}$$

Thus we can see the interrelation of time, decoherence, and the elimination of the uncertainty relations. The usual procedure is to postulate the existence of time and follow the chain time $\rightarrow$ decoherence $\rightarrow$ elimination of the uncertainty relations. In this case we are using the traditional way of thinking of classical and nonrelativistic quantum mechanics: *time is a primitive concept*. But, as we have explained, another chain is feasible: decoherence $\rightarrow$ elimination of the uncertainty relations $\rightarrow$ time, and that would be the chain that we must use if we consider the whole universe. In this last case we could postulate (based on obvious observational fact) that the quantum physics of the universe is such that it has a tendency toward the classical regime (decoherence + elimination of the uncertainty relations). This tendency would be the primitive concept in this case and *time would be a derived concept* in perfect accord with Mach's philosophy [20] (see also ref. 21). So question (iii) of the introduction is answered. We will come back to these arguments in the cosmological case in Section 4.4.3, where we will go further. We will postulate the existence of a parameter η that at the quantum level will take the role of time, namely $|\eta\rangle = e^{-iHt}|0\rangle$, and show that there is decoherence in this parameter, which therefore becomes a classical one.

[12] This will be the case in the first problem. In the second one we will deal with the real universe and reproduce the same kind of argument.

3.4.2. Why the Outgoing Solutions?

The answer to this question can be found in one of the essential properties of the actual state of the real universe: *it is time-asymmetric* in spite of the fact that its evolution laws are time-symmetric:

(i) Even if we have incoming and outgoing scattering states, only the second ones behave spontaneously, while the first ones must be produced by a source of energy [22].[13]

(ii) Even if we have advanced and retarded solutions of the Maxwell equations, the time asymmetry of the state of the universe forces us to use the latter ones and to neglect the former ones.

(iii) At the quantum level, the space of admissible solutions is not the whole Hilbert space, but a subspace of this space, where causality can be introduced.[14]

Namely, as the laws of physics are time symmetric, they always give us two *t*-symmetric possibilities. The universe is *t*-asymmetric and corresponds to one of these possibilities.

From what we have said it must be clear that, in order to be completely satisfactory, the choice must be done in the whole universe, namely it must be global [22]. Let us only consider case (i); we must only choose outgoing states answering question (iv).

For our first problem we may use the WKB solution of the equation

$$\frac{d^2\psi(x)}{dx^2} + k^2(x)\psi(x) = 0 \tag{47}$$

which is

$$\psi(x) = [k(x)]^{-1/2} \exp\left[\pm i \int k(x)\, dx\right] \tag{48}$$

Then if we write the solutions of Eq. (2) as

$$\psi_{\omega lm}(\mathbf{x}) = \psi_{\omega lm}(r, \theta, \varphi) = Y_l^m(\theta, \varphi)\, \frac{u_{\omega l}(r)}{r} \tag{49}$$

the function $u_{\omega l}(r)$ satisfies the equation

[13] Incoming states will transform the stable nucleus into an unstable one. Outgoing states will radiate the energy produced when the unstable nucleus evolves to a equilibrium state in a spontaneous way.

[14] As causality is related to the analytic properties of wave functions, when we promote the energy ω to a complex variable z, we can choose the admissible solutions as those wave functions that are analytic and bounded in the lower complex half-plane of variable z, even if we could choose those that have the same properties in the upper plane (or, more precisely, such that they belong to the Hardy class function space from below (or above) [27]. In this way causality is introduced along with its physical consequences: dispersion relation, the fluctuation-dissipation theorem, the growth of entropy, etc. [17, 19].

$$\frac{d^2 u_{\omega l}(r)}{dr^2} + 2M\left[\omega - \frac{l(l+1)}{2Mr^2} - W(r)\right]u_{\omega l}(r) = 0 \tag{50}$$

If we write

$$k^2_{\omega l}(r) = 2M\left[\omega - \frac{l(l+1)}{2Mr^2} - W(r)\right] \tag{51}$$

the WKB solution is

$$u_{\omega l}(r) = [k_{\omega l}(r)]^{-1/2} \exp\left[\pm i \int_0^r k_{\omega l}(r)\, dr\right] \tag{52}$$

These are the two possibilities produced by our time-symmetric theory. Then, in order for solution (52) to be outgoing, we must choose the sign $+$. In fact, far from the nucleus, when $t \to \infty$, $r \to \infty$, we have $k_{\omega l}(r) \to k_\omega = +\sqrt{2M\omega}$ and considering the time evolution factor, we have

$$e^{-iHt} u_{\omega l}(r) = [k_\omega]^{-1/2} \exp[-i(\omega t - k_\omega r + \text{const})] \tag{53}$$

and the wave evolution becomes the outgoing unilateral shift:

$$r = \frac{\omega}{k_\omega} t + \text{const} \tag{54}$$

So question (iv) of the Introduction is also answered.[15]

4. THE COSMOLOGICAL PROBLEM

4.1. The Model

Let us consider the flat Roberson–Walker universe [23, 24] with a metric

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2) \tag{55}$$

where η is the conformal time and a the scale of the universe. Let us consider

[15] It is interesting to observe that, following the ideas of ref. 22, this typical outgoing nature of the shift (which is extremely important in the cosmological models [22]) is only possible because from the very beginning the configuration space has a characteristic structure: it is spherically symmetric and this fact *defines the asymmetry* $r > 0$, namely the asymmetry that says that the origin O is substantially different than the sphere at the infinity. Therefore: *it is an asymmetry in configuration space that really introduces time asymmetry* [20]. Without this asymmetry all the treatment would be impossible since the "outgoing" notion itself would be meaningless, and the choice of the lower half-plane unmotivated (at least in the case where we consider our system as the whole universe and therefore we have no other physical phenomena to play with). Therefore the fact that an isolated nucleus radiates and never receives spontaneously radiation from the exterior is a consequence of the fundamental time asymmetry of the universe. But if, as a theoretical example, we consider this isolated system as the whole universe, it is a consequence of the asymmetry of the configuration space of the model, in complete agreement with ref. 20. In the second example the time asymmetry has more or less the same origin: the global asymmetry of the universe, as explained in ref. 22.

a free neutral scalar field Φ and let us couple this field to the metric with a conformal coupling ($\xi = 1/6$). The total action reads $S = S_g + S_f + S_i$ and the gravitational action is

$$S_g = M^2 \int d\eta \left[-\frac{1}{2} \dot{a}^2 - V(a) \right] \tag{56}$$

where M is the Planck mass, $\dot{a} = da/d\eta$, and the potential V contains the cosmological constant term and eventually the contribution of some form of classical matter. We suppose that V has a bounded support $0 \leq a \leq a_1$. We expand the field Φ as

$$\Phi(\eta, \mathbf{x}) = \int f_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \, d\mathbf{k} \tag{57}$$

where the components of $\mathbf{k}$ are three continuous variables.

The Wheeler–DeWitt equation for this model reads [compare with Eq. (1) for the first example]

$$H\Psi(a, \Phi) = (h_g + h_f + h_i)\Psi(a, \Phi) = 0 \tag{58}$$

where

$$\begin{aligned} h_g &= \frac{1}{2M^2} \partial_a^2 + M^2 V(a) \\ h_f &= -\frac{1}{2} \int (\partial_{\mathbf{k}}^2 - k^2 f_{\mathbf{k}}^2) \, d\mathbf{k} \\ h_i &= \frac{1}{2} m^2 a^2 \int f_{\mathbf{k}}^2 \, d\mathbf{k} \end{aligned} \tag{59}$$

where m is the mass of the scalar field, $\mathbf{k}/a$ is the linear momentum of the field, and $\partial_{\mathbf{k}} = \partial/\partial f_{\mathbf{k}}$.

We can now go to the semiclassical regime using the WKB method [25], writing $\Psi(a, \Phi)$ as

$$\Psi(a, \Phi) = \exp[iM^2 S(a)]\chi(a, \Phi) \tag{60}$$

and expanding S and χ as

$$S = S_0 + M^{-1} S_1 + \ldots, \qquad \chi = \chi_0 + M^{-1}\chi_1 + \ldots \tag{61}$$

To satisfy Eq. (58) at the order M^2, the principal Jacobi function $S(a)$ must satisfy the Hamilton–Jacobi equation

$$\left(\frac{dS}{da}\right)^2 = 2V(a) \tag{62}$$

We can now define the (semi)classical time as in Section 3.3 (but now using an approximate solution only). It is a parameter $\eta = \eta(a)$ such that

$$\frac{d}{d\eta} = \frac{dS}{da}\frac{d}{da} = \pm\sqrt{2V(a)}\,\frac{d}{da} \tag{63}$$

So

$$\eta = \frac{1}{\sqrt{2}}\int_{a_0}^{a}\frac{1}{\sqrt{V(a)}}\,da \tag{64}$$

which must be compared with Eq. (46), but in this equation the trajectories are completely classical, while in Eq. (64) only a is classical and Φ remains a quantum variable. The solution of Eq. (64) is $a = \pm F(\eta, C)$, where C is an arbitrary integration constant. Different values of this constant and of the $\pm$ sign give different classical solutions for the geometry.

Then, in the next order of the WKB expansion, the Schrödinger equation reads

$$i\,\frac{d\chi}{d\eta} = h(\eta)\chi \tag{65}$$

where

$$h(\eta) = h_f + h_i(a) \tag{66}$$

Precisely,

$$h(\eta) = -\frac{1}{2}\int\left[-\frac{\partial^2}{\partial f_{\mathbf{k}}^2} + \Omega_{\mathbf{k}}^2(a)\,f_k^2\right]d\mathbf{k} \tag{67}$$

where

$$\Omega_{\varpi}^2 = m^2a^2 + k^2 = m^2a^2 + \varpi \tag{68}$$

and $\varpi = k^2$ and $k = |\mathbf{k}|$. So the time dependence of the Hamiltonian comes from the function $a = a(\eta)$.

Let us now consider a scale of the universe such that $a_{\text{out}} \gg a_1$. We will consider the evolution in this region where the geometry is almost constant. Therefore we have an adiabatic final vacuum $|0\rangle$ and adiabatic creation and annihilation operators $a_{\mathbf{k}}^\dagger$ and $a_{\mathbf{k}}$. Then $h = h(a_{\text{out}})$ reads

$$h = \int \Omega_{\varpi} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} \, d\mathbf{k} \tag{69}$$

We can now consider the Fock space and a basis of vectors

$$|\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n, \ldots\rangle \cong |\{k\}\rangle = a_{\mathbf{k}_1}^{\dagger} a_{\mathbf{k}_2}^{\dagger} \ldots a_{\mathbf{k}_n}^{\dagger} \ldots |0\rangle \tag{70}$$

where we have called $\{k\}$ the set $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n, \ldots$. The vectors of this basis are eigenvectors of h:

$$h|\{k\}\rangle = \omega|\{k\}\rangle \tag{71}$$

where

$$\omega = \sum_{\mathbf{k} \in \{\mathbf{k}\}} \Omega_{\varpi} = \sum_{\mathbf{k} \in \{\mathbf{k}\}} (m^2 a_{\text{out}}^2 + \varpi)^{1/2} \tag{72}$$

We can now use this energy to label the eigenvector as

$$|\{k\}\rangle = |\omega, [\mathbf{k}]\rangle \tag{73}$$

where $[\mathbf{k}]$ is the remaining set of labels necessary to define the vector unambiguously. $\{|\omega, [\mathbf{k}]\rangle\}$ is obviously an orthonormal basis, so Eq. (69) reads

$$h = \int \omega|\omega, [\mathbf{k}]\rangle\langle\omega, [\mathbf{k}]| \, d\omega \, d[\mathbf{k}] \tag{74}$$

This is the Hamiltonian that corresponds to (4) in the cosmological case.

4.2. Decoherence in the Other Dynamical Variables

We can obtain the decoherence in the energy as in Section 2. Then, if we reintroduce the other dynamical variables in Eq. (12), we obtain

$$(\rho_*| = \int \rho_{\omega[\mathbf{k}][\mathbf{k}']}(\omega, [\mathbf{k}], [\mathbf{k}']| \, d\omega \, d[\mathbf{k}] \, \mathbf{d}[\mathbf{k}'] \tag{75}$$

where $\{(\omega, [\mathbf{k}], [\mathbf{k}']|, (\omega, \omega', [\mathbf{k}], [\mathbf{k}']\}$ is the cobasis $\{(\omega|, (\omega, \omega'|\}$, but now showing the hidden $[\mathbf{k}]$. This equation corresponds to (14) in the cosmological case.

Let us observe that if we use polar coordinates for $\mathbf{k}$, Eq. (57) reads

$$\Phi(x, n) = \int \sum_{lm} \phi_{klm} \, dk \tag{76}$$

where

$$\phi_{klm} = f_{k,l}(\eta, r) Y_m^l(\theta, \varphi) \tag{77}$$

where k is a continuous variable; $l = 0, 1, \ldots$; $m = -l, \ldots, l$; and Y are spherical harmonic functions. So the indices k, l, m contained in the symbol $\mathbf{k}$ are partially discrete and partially continuous.

As $\rho_*^\dagger = \rho_*$, then $\rho^*_{\omega[\mathbf{k}'][\mathbf{k}]} = \rho_{\omega[\mathbf{k}][\mathbf{k}']}$ and therefore a set of vectors $\{|\omega, [\mathbf{l}]\rangle\}$ exists such that

$$\int \rho_{\omega[\mathbf{k}][\mathbf{k}']}|\omega, [\mathbf{l}]\rangle_{[\mathbf{k}']}\, d[\mathbf{k}'] = \rho_{\omega[\mathbf{l}]}|\omega, [\mathbf{l}]\rangle_{[\mathbf{k}]} \tag{78}$$

namely $\{|\omega, [\mathbf{l}]\rangle\}$ is the eigenbasis of the operator $\rho_{\omega[\mathbf{k}][\mathbf{k}']}$. Then $\rho_{\omega[\mathbf{l}]}$ can be considered as an ordinary diagonal matrix in discrete indices like l and m, and a generalized diagonal matrix in continuous indices like k.[16] Under the diagonalization process, Eq. (75) is written as

$$(\rho_*| = \int U^{\dagger[\mathbf{l}]}_{[\mathbf{k}]}\rho_{\omega[\mathbf{l}][\mathbf{l}']}U^{[\mathbf{l}']}_{[\mathbf{k}']} \times U^{\dagger[\mathbf{l}'']}_{[\mathbf{k}']}(\omega, [\mathbf{l}''], [\mathbf{l}''']|U^{[\mathbf{l}''']}_{[\mathbf{k}]}\, d\omega\, d[\mathbf{k}]\, \mathbf{d}[\mathbf{k}']\, d[\mathbf{l}]\, \mathbf{d}[\mathbf{l}']\, d[\mathbf{l}'']\, \mathbf{d}[\mathbf{l}'''] \tag{79}$$

where $U^{\dagger[\mathbf{l}]}_{[\mathbf{k}]}$ is the unitary matrix used to perform the diagonalization and

$$\rho_{\omega[\mathbf{l}][\mathbf{l}']} = \rho_{\omega[\mathbf{l}]}\delta_{[\mathbf{l}][\mathbf{l}']} \tag{80}$$

where

$$\rho_{\omega[\mathbf{l}][\mathbf{l}]} = \rho_{\omega[\mathbf{l}]} = \int U^{[\mathbf{k}]}_{[\mathbf{l}]}\rho_{\omega[\mathbf{k}][\mathbf{k}']}U^{\dagger[\mathbf{k}']}_{[\mathbf{l}]}\, d[\mathbf{k}]\, d[\mathbf{k}'] \tag{81}$$

so we can define

$$(\omega, [\mathbf{l}]| = (\omega, [\mathbf{l}], [\mathbf{l}]| = \int U^{[\mathbf{k}]}_{[\mathbf{l}]}(\omega, [\mathbf{k}], [\mathbf{k}']|U^{\dagger[\mathbf{k}']\dagger}_{[\mathbf{l}]}\, d[\mathbf{k}]\, d[\mathbf{k}'] \tag{82}$$

We can repeat the procedure with vectors $(\omega, \omega', [\mathbf{k}], [\mathbf{k}']|$ and obtain the vector $(\omega, \omega', [\mathbf{l}]|$. In this way we obtain a diagonalized cobasis $\{(\omega, [\mathbf{l}]|, (\omega, \omega', [\mathbf{l}]|\}$. So we can now write the equilibrium state as

$$\rho_* = \int \rho_{\omega[\mathbf{l}]}(\omega, [\mathbf{l}]|\, d\omega\, d[\mathbf{l}] \tag{83}$$

which corresponds to (16) in the cosmological case. Since vectors $(\omega, [\mathbf{l}]|$ can be considered as diagonals in all the variables, we have obtained decoherence in all the dynamical variables. This fact will become clearer once we

[16] E.g., we can deal with this generalized matrix by rigging the space $\mathcal{S}$ and using the Gel'fand–Maurin theorem [26]; this procedure allows us to define a generalized state eigenbasis for system with continuous spectrum. It has been used to diagonalize Hamiltonians with continuous spectra in refs. 17, 27, 28, etc.

study the observables related to this vector and introduce the notion of *final pointer basis*.

Let us consider the observable basis $\{|\omega, [\mathbf{l}]), |\omega, \omega', [\mathbf{l}])\}$ dual to the state cobasis $\{(\omega, [\mathbf{l}]|, (\omega, \omega', [\mathbf{l}]|\}$. From Eq. (6) and as the ω does not play any role in the diagonalization procedure, we obtain

$$|\omega, [\mathbf{l}]) = |\omega, [\mathbf{l}]\rangle\langle\omega, [\mathbf{l}]|, \qquad |\omega, \omega', [\mathbf{l}]) = |\omega, [\mathbf{l}]\rangle\langle\omega', [\mathbf{l}]| \tag{84}$$

So in the basis $\{|\omega, [\mathbf{l}]), |\omega, \omega', [\mathbf{l}])\}$ the Hamiltonian reads

$$h = \int \omega|\omega, [\mathbf{l}])\, d\omega\, d[\mathbf{l}] = \int \omega|\omega, [\mathbf{l}]\rangle\langle\omega, [\mathbf{l}]|\, d\omega\, d[\mathbf{l}] \tag{85}$$

Now, we can also define the operators

$$\mathbf{L} = \int \mathbf{l}|\omega, [\mathbf{l}])\, d\omega\, d[\mathbf{l}] = \int \mathbf{l}|\omega, [\mathbf{l}]\rangle\langle\omega, [\mathbf{l}]|\, d\omega\, d[\mathbf{l}] \tag{86}$$

which can also be written, as in Eq. (17),

$$L_i = \int l_i|\omega, [\mathbf{l}])\, d\omega\, d[\mathbf{l}] = \int l_i|\omega, [\mathbf{l}]\rangle\langle\omega, [\mathbf{l}]|\, d\omega\, d[\mathbf{l}] \tag{87}$$

where i is an index such that it covers all the dimension of the $\mathbf{l}$.[17] Now we can consider the set (h, L_i), which is a CSCO, since all the members of the set commute because they share a common basis, and find the corresponding eigenbasis of the set, precise $|\omega, [\mathbf{l}]\rangle$, since[18]

$$h|\omega, [\mathbf{l}]\rangle = \omega|\omega, [\mathbf{l}]\rangle \tag{88}$$

$$L_i|\omega, [\mathbf{l}]\rangle = l_i|\omega, [\mathbf{l}]\rangle \tag{89}$$

Of course the L_i are constants of the motion because they commute with h. From all these equations we can say that:

(i) (h, L_i) is the final pointer CSCO.

(ii) $\{|\omega, [\mathbf{l}]), |\omega, \omega', [\mathbf{l}])\}$ is the final pointer observable basis.

(iii) $\{(\omega, [\mathbf{l}]|, (\omega, \omega', [\mathbf{l}]|\}$ is the final pointer states cobasis.

In fact, from Eq. (83) we see that the final equilibrium states has only diagonal terms in this state (those corresponding to vector $(\omega, [\mathbf{l}|)$, it has no off-diagonal terms (those corresponding to vectors $(\omega, \omega', [\mathbf{l}]|$, $(\omega, [\mathbf{k}], [\mathbf{k}']|$, or $(\omega, \omega', [\mathbf{k}], [\mathbf{k}']|)$, and therefore we have decoherence in all the dynamical variables.

[17] In principle the matter field Φ may have any number of particles N. But since we are working in the final stage of the universe evolution with $a \sim a_{\text{out}}$, this number is a constant. Then the number of observables in the CSCO is $4N$ and the ket in configuration variables read $|\eta, [\mathbf{x}])\rangle = |\{x\}\rangle$, where $[\mathbf{x}]$ is the space position of the $4N$ particles.

[18] On some occasions we will call $h = L_0$ and $\omega = l_0$.

4.3. The Limit $\hbar \to 0$ and the Classical $\rho_*^{(cl)}(x, k)$

Let us restore the notation $\{l\} = (\omega, [\mathbf{l}], \{k\} = (\omega, [\mathbf{k}])$, as in Eq. (73), and let us consider the configuration kets $|\{x\}\rangle = |\eta, [\mathbf{x}]\rangle$. Since we are considering the period when $a \sim a_{\text{out}}$ the system with Hamiltonian (67) is just a set of infinite oscillators with constants $\Omega_{\mathbf{k}}(a_{\text{out}})$ that represent a scalar field with mass ma_{out}. Then we are just dealing with a classical set of N particles with coordinates $[\mathbf{x}]$ and momenta $[\mathbf{k}]$. Then, as in Eq. (33), we can introduce the Wigner function correponding to generalized state $|\{l\})$:

$$\rho_{\{l\}}^{(W)}([\mathbf{x}], [\mathbf{k}]) = \pi^{-4N} \int (\{l\}|\mathbf{x} + \lambda\rangle\langle\mathbf{x} - \lambda|) e^{2i[\lambda]\cdot[\mathbf{k}]}\, d^{4n}\, \lambda \tag{90}$$

Using the same reasoning used to obtain Eq. (36), we obtain

$$\rho_{\{l\}}^{(W)}([\mathbf{x}], [\mathbf{k}]) = \prod_i \delta(L_i^W([\mathbf{x}], [\mathbf{k}]) - l_i) \tag{91}$$

where $L_i^W([\mathbf{x}], [\mathbf{k}])$ is the classical observable obtained from L_i via the Wigner integral (considering $h = L_0$ and including 0 among the indices i). Now, with the new notation (90), Eq. (88) reads

$$\rho_* = \int \rho_*\{l\}(\{l\}|\, d\{l\} \tag{92}$$

Then if we write

$$\rho_*^{(W)}([\mathbf{x}], [\mathbf{k}]) = \pi^{-4N} \int (\{\rho_*|\mathbf{x} + \lambda\rangle\langle\mathbf{x} - \lambda|) e^{2i[\lambda]\cdot[\mathbf{k}]}\, d^{4n}\lambda \tag{93}$$

we obtain, as in Eq. (31),

$$\rho_*^{(W)}([\mathbf{x}], [\mathbf{k}]) = \rho_*^{(W)}(L_0^W([\mathbf{x}], [\mathbf{k}]), L_1^W([\mathbf{x}], [\mathbf{k}]), \ldots) \tag{94}$$

So finally

$$\begin{aligned}\rho_*^{(W)}([\mathbf{x}], [\mathbf{k}]) &\sim \int d\{l\}\ \rho_*^{(W)}([\mathbf{x}], [\mathbf{k}])\delta(\{L^W\} - \{l\}) \\ &= \int d\{l\}\ \rho_{\{l\}}|\prod_i \delta(L_i^W - l_i)\end{aligned} \tag{95}$$

The last equation can be interpreted as follows:

(i) $\delta(\{p\} - \{l\})$ is a classical density function, strongly peaked at certain values of the constants of motion $\{l\}$, corresponding to a set of trajectories where the momenta are equal to the eigenvalues of Eqs. (88) and (89), namely $L_i^W = l_i$ $(i = 0, 1, 2, \ldots)$.

(ii) $\rho_{\{l\}}$ is the probability to be in one of the sets of trajectories labeled by $\{l\}$. Precisely: if some initial density matrix is given, from Eq. (83) it is evident that its diagonal terms $\rho_{\{l\}}$ are the probabilities to be in the states $(\omega, [\mathbf{l}]|$ and therefore the probability to find, in the corresponding classical equilibrium density function $\rho_*^{(W)}([\mathbf{x}], [\mathbf{k}])$, the density function $\delta(\{L^W\} - \{l\})$, namely the probability of the set of trajectories labeled by $\{\mathrm{l}\} = (\omega, [\mathbf{l}])$.

(iii) As in Eq. (40), let $\mathbf{a}$ be the coordinate classically conjugate to $\mathbf{l}$ and let $\mathbf{a}_0$ be the coordinate $\mathbf{a}$ at time $\eta = 0$; then we obtain the classical trajectories:

$$\mathbf{a} = \mathbf{l}\eta + \mathbf{a}_0 \tag{96}$$

(iv) Let us now write $\rho_*\{l\} = p_{\{l\}[\mathbf{a}_0]}$. Actually $p_{\{l\}[\mathbf{a}_0]}$ is not a function of $\mathbf{a}_0$, it simply is a constant in $\mathbf{a}_0$, since $\mathbf{a}_0$ is only an arbitrary point and our model is spatially homogeneous. Then we can write

$$p_{\{l\}[\mathbf{a}_0]} = \int p_{\{l\}[\mathbf{a}_0]} \prod_{i=1} \delta(a_i - a_{0i})\, d[\mathbf{a}_0] \tag{97}$$

In this way we have changed the role of $\mathbf{a}_0$; it was a fixed (but arbitrary) point and it is how a variable that moves all over the space. Then Eq. (95) reads

$$\rho_*^{(\mathrm{cl})}([\mathbf{x}], [\mathbf{k}]) \sim \int p_{\{l\}[\mathbf{a}_0]} \prod_i \delta(L_i^{\mathrm{W}} - l_i) \prod_{j=1} \delta(a_j - a_{0j})\, d[\mathbf{a}_0]\, d\{l\} \tag{98}$$

So if we write

$$\rho^{(\mathrm{cl})}_{\{l\}[\mathbf{a}_0]}([\mathbf{x}], [\mathbf{k}]) = \prod_{i=0} \delta(L_i^W - l_i) \prod_{j=1} \delta(a_j - a_{0j}) \tag{99}$$

we have

$$\rho_*^{(\mathrm{cl})}([\mathbf{x}], [\mathbf{k}]) \sim \int p_{\{l\}[\mathbf{a}_0]} \rho^{(\mathrm{cl})}_{\{l\}[\mathbf{a}_0]}([\mathbf{x}], [\mathbf{k}])\, d[\mathbf{a}_0]\, d\{l\} \tag{100}$$

From Eq. (99) we see that $\rho^{(\mathrm{cl})}_{\{l\}[\mathbf{a}_0]}([\mathbf{x}], [\mathbf{k}]) \neq 0$ only in a narrow strip around the classical trajectory (96) defined by the momenta $\{l\}$ and passing through the point $[\mathbf{a}_0]$ [actually, the density function is as peaked as is allowed by the uncertainty principle, so its width is essentially $O(\hbar)$, since the δ-functions of all the equation are really Dirac deltas only when $\hbar \to 0$]. So we have proved Eq. (100), which, in fact, is Eq. (3) as announced.[19]

Then we have obtained the classical limit. When $\eta \to \infty$ the quantum density ρ becomes a diagonal density matrix ρ_*. The corresponding classical distribution $\rho_*^{(cl)}([\mathbf{x}], [\mathbf{k}])$ can be expanded as a sum of classical trajectory density functions $\rho_{\{l\}[\mathbf{a}_0]}^{(cl)}([\mathbf{x}], [\mathbf{k}])$, each one weighted by its corresponding probability $p_{\{l\}[\mathbf{a}_0]}$. So, as the limit of our quantum model we have obtained a statistical classical mechanical model, and the classical realm appears.

4.4. Localization and Correlations

Under appropriate initial conditions the motion can be concentrated in just one trajectory, showing the presence of correlations in this trajectory [4]. The evolution of the concentration depends on potential $V(a)$. Of course, our "trajectories" are not only one trajectory for a one-particle state, but N trajectories (each one corresponding to a momentum $(l_1, l_2, \ldots, l_n) = \{l\}$ and passing by a point $(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) = [\mathbf{a}]$) for the n-particle states.

4.5. Discussion and Comments

4.5.1. Characteristic Times

The decaying term of Eq. (10) (i.e., the second term on the r.h.s.) can be analytically continued using the techniques explained in refs. 9, 17, and 24. In these papers it is shown that each pole $z_i = \omega_i - i\gamma_i$ of the S-matrix (corresponding to the evolution $a_{\text{in}} \to a_{\text{out}}$ [24]) of the problem considered originates a damping factor $e^{-\gamma_i \eta}$. Then if $\gamma = \min(\gamma_i)$, the characteristic decoherence time is γ^{-1}. This computation is done in the specific models of refs. 24. If $\gamma \ll 1$, even if the Riemann–Lebesgue theorem is always valid, there is no practical decoherence since $\gamma^{-1} \gg 1$.

[19] In this section, as in Section 2.2, we are faced with the following problem: $\rho_*^{(cl)}([\mathbf{x}], [\mathbf{k}])$ is a constant that we want to decompose into functions $\rho_{\{l\}[\mathbf{a}_0]}^{(cl)}([\mathbf{x}], [\mathbf{k}])$ which are different from zero only around the trajectory (96) and therefore are variables in $\mathbf{a}$. Then, essentially we use the fact that if $f(x, y) = g(y)$ is a constant function in x, we can decompose it as

$$g(y) = \int g(y)\delta(x - x_0)\, dx_0$$

namely the densities $\delta(x - x_0)$ are peaked in the trajectories $x = x_0 = \text{const}$, $y = \text{var}$, and therefore are functions of x. This trajectories play the role of those of Eq. (97). As all the physics, including the correlations, is already contained in Eq. (95), the reader may just consider the final part of this section, from Eq. (97) to Eq. (100), a didactical trick.

4.5.2. Sets of Decoherent Trajectories

It is usual to say that in the classical regime there is decoherence of the set trajectories labeled by the constant of the motion ω, [**l**]. This result can easily be obtained with our method in the following way.

(i) Let us consider two different states $|\omega[\mathbf{l}]\rangle$ and $|\omega'[\mathbf{l}']\rangle$ that will define classes of trajectories with different constants of the motion $(\omega, [\mathbf{l}]) \neq (\omega', [\mathbf{l}'])$. We must compute

$$\langle\omega[\mathbf{l}]|\rho_*|\omega'[\mathbf{l}']\rangle = (\rho_*\|\omega\omega'[\mathbf{l}][\mathbf{l}'])$$
$$= \left[\int \rho_{\omega''[\mathbf{l}'']}(\omega''[\mathbf{l}'']|\, d\omega''\, d[\mathbf{l}'']\right]|\omega\omega'[\mathbf{l}][\mathbf{l}']) = 0 \quad (101)$$

due to the orthogonality of the basis $\{(\omega, [\mathbf{l}]|, (\omega, \omega', [\mathbf{l}]|\}$.

(ii) But if we compute

$$\langle\omega[\mathbf{l}]|\rho_*|\omega[\mathbf{l}]\rangle = (\rho_*\|\omega[\mathbf{l}]) = \left[\int \rho_{\omega''[\mathbf{l}'']}(\omega''[\mathbf{l}'']|\, d\omega''\, d[\mathbf{l}'']\right]|\omega[\mathbf{l}])$$
$$= \int \rho_{\omega''[\mathbf{l}'']}\delta(\omega - \omega'')\delta([\mathbf{l}] - [\mathbf{l}''])\, d\omega''\, d[\mathbf{l}'']$$
$$= \rho_{\omega[\mathbf{l}]} \neq 0 \quad (102)$$

The last two equations complete the demonstration. We will discuss the problem of the decoherence of two trajectories with the same $\{l\}$ but different $[\mathbf{a}_0]$ in Section 4.5.4.

4.5.3. A Discussion on Time Decoherence

It is well known that one of the main problems of quantum gravity is the problem of defining the time [29]. A poorly studied feature of this problem is that there must be a decoherence process related to time since time is treated as a classical variable. In this subsection, using the functional technique, we will give a model that shows that this is the case (but we must emphasize that this subject is not completely developed).

Let us postulate that there is a parameter η such that the quantum states evolve as[20]

$$|\eta\rangle = e^{-ih\eta}|0\rangle \quad (103)$$

We must compute $\langle\eta|\rho_*|\eta'\rangle$, where $|\eta\rangle$ and $|\eta'\rangle$ are two states of the

[20] Of course η is the conformal time of Eq. (65), since (103) is a consequence of (65). But now we have postulated this last equation and we are searching for the quantum properties of η.

system for different times. We do not know if $\langle\eta|\rho_*|\eta'\rangle$ will decohere or not. If it decoheres, we can say that the parameter η is classical. $|\eta\rangle\langle\eta'|$ can be considered as an observable; then

$$\langle\eta'|\rho_*|\eta\rangle = (\rho_*\|\eta\rangle\langle\eta'|) \tag{104}$$

But

$$(\omega\|\eta\rangle\langle\eta'|) = (\omega|e^{-ih\eta}|0\rangle\langle 0|e^{ih\eta'}) = [e^{ih\eta'}(\omega|e^{-ih\eta}]\|0\rangle\langle 0|) \tag{105}$$

Now, for any observable O we have

$$\begin{aligned}
[e^{ih\eta'}(\omega|e^{-ih\eta}]\|O) &= [e^{ih\eta'}(\omega|e^{-ih\eta}]\bigg|\bigg[\int O_{\omega'}|\omega')\,d\omega' \\
&\quad + \iint O_{\omega'\omega''}|\omega',\omega'')\,d\omega'\,d\omega''\bigg) \\
&= [e^{ih\eta'}(\omega|e^{-ih\eta}]\bigg|\bigg[\int O_{\omega'}|\omega')\,d\omega' + \ldots \\
&= (\omega|\bigg[\int O_{\omega'}e^{-i\omega'\eta}|\omega')e^{i\omega'\eta'}\,d\omega'\bigg]\bigg) \\
&= e^{-i\omega(\eta'-\eta)}(\omega|O)
\end{aligned} \tag{106}$$

where the second term disappears since $(\omega|\omega',\omega'') = 0$. Thus

$$(\omega\|\eta\rangle\langle\eta'|) = e^{-i\omega(\eta'-\eta)}(\omega\|0\rangle\langle 0|) \tag{107}$$

So now we can compute the following two cases:

(i)

$$\begin{aligned}
\langle\eta'|\rho_*|\eta\rangle = (\rho_*\|\eta\rangle\langle\eta'|) &= \bigg[\int \rho_\omega(\omega|\,d\omega\bigg]\|\eta\rangle\langle\eta'|) \\
&= \int \rho_\omega e^{-i\omega(\eta'-\eta)}(\omega\|0\rangle\langle 0|)\,d\omega \to 0
\end{aligned} \tag{108}$$

when $|\eta'-\eta| \to \infty$, due to the Riemann–Lebesgue theorem.

(ii) Analogously,

$$\langle\eta|\rho_*|\eta\rangle = \int \rho_\omega(\omega\|0\rangle\langle 0|)\,d\omega \neq 0 \tag{109}$$

So we have time decoherence for two times η and η' if they are far enough apart.

This result is important for the problem of time definition, since in order to have a reasonable classical time this variable must first decohere. The above result shows that this is the case for η and η' far enough apart,[21] but also that for closer times (namely such that their difference is smaller than the Planck time) there is no decoherence and time cannot be considered as a classical variable. Classical time is a familiar concept, but the nature of nondecohered quantum time is open to discussion. We remark that we have followed the second line of thought of Section 2.3.1: we have supposed the existence of an evolution $e^{-ih\eta}$, where η is only a parameter. We have proved that decoherence appears and find that η behaves like a classical variable. Maybe this is the better way to introduce the classical time: to postulate a "quantum" η and find its properties. Moreover, we have proved that the second approach of Section 2.3.1 can also be followed.

4.5.4. Decoherence in the Space Variables

Now that we know that there is time decoherence, we can repeat the reasoning for the rest of the variables **a** at time $\eta = 0$ and change Eq. (103) to

$$|[\mathbf{a}]\rangle = e^{i[\mathbf{a}]\cdot[\mathbf{I}]}|\mathbf{0}\rangle \tag{110}$$

We will reach the following conclusions:

(i)

$$\langle[\mathbf{a}]|\rho_*|[\mathbf{a}']\rangle \to 0 \tag{111}$$

when $|\mathbf{a} - \mathbf{a}'| \to \infty$.

(ii)

$$\langle[\mathbf{a}]|\rho_*|[\mathbf{a}]\rangle \neq 0 \tag{112}$$

Therefore there is also decoherence between two trajectories with the same $\{l\}$ but different $[\mathbf{a}_0]$.

These facts complete the scenario about decoherence and the final classical limit. An analogy of Section 3.3.2 in the cosmological case can be found in ref. 21.

5. CONCLUSION

We are convinced that the method of refs. 9 and 10 is the best way to study both the Mott and the cosmological problem and to find analogies and differences between them. We hope that the reader will share our conviction.

[21] Using the method of Section 4.3.1, we can compute γ. Decoherence will take place for $|\eta - \eta'| > \gamma^{-1}$.

For the important cosmological problem we have shown that after the WKB expansion and decoherence and the final classical limit process, (i) our quantum model has a defined classical time η and a defined classical geometry related by Eq. (65); (ii) decoherence has appeared in a well-defined final pointer basis; and (iii) the quantum field has originated a classical final distribution function [Eq. (100)] that is a weighted average of some set densities, each one related to a classical trajectory. The weight coefficients are the probabilities of each trajectory.

We can see that if instead of a spinless field we coupled the geometry to a spin-2 metric fluctuation field, the result would be more or less the same. Then the corresponding quantum fluctuations would become classical fluctuations that would correspond to matter inhomogeneities (galaxies, clusters of galaxies, etc.) that will move along the trajectories described above. This subject will be treated elsewhere in greater detail.

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